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Perturbation of Sets and Centers

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Abstract. Given a bounded set A, a Chebyshev center (when it exists) is–in some sense–a candidate to give a global information on the set. Finding the centers of A is of great importance for applications. In many cases, it is very important to understand how they change when the set A is perturbed. Our main result is a new characterization of Hilbert spaces: in fact, we will show that the best estimate we can give in these spaces, concerning perturbations of sets, cannot be expected outside this class of spaces. Moreover, we collect, we partly sharpen and we reprove in a simple way most known results.

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1. Introduction and Definitions

Let A be a bounded set. If we want to substitute A by a singleton, a Chebyshev center (center, for short) may represent the whole set, in the sense that no point of A is too far from a center. Finding the centers of a set is important, also in view of applications. For example, one of the most important problems considered in location theory is the following: given a set of clients, find the best location for a service center (a facility serving a set of customers). In case it is important to locate the facility not too far from all clients, the best location should be a center; in other terms, in this case, given a set A we look for a point c minimizing the largest distance from the points of A.

Sometimes the set A (that is, the set of clients), is not well known, or it is subject to changes; so it becomes interesting to know how centers are sensitive to motion of clients. Consider the following example: assume that the service must be provided by a mobile facility; a network maintainer should reach all customers in some area and the customers move with their devices (cellular phones, for example).

We assume, as often done, that the distance is measured by a norm; so we consider a Banach space X, and a bounded subset A; also, we do the general assumption that A contains at least two elements, otherwise most facts become trivial. Moreover, we assume that A is closed; recall that A and its closed convex hull have the same Chebyshev centers and radius.

Finding a center amounts to solve a minimax problem: we must minimize (on the whole space X) a function depending on the set A.

We sometimes assume that A is a finite set; most results obtained for these sets extend to compact sets by using an *ε-net* argument.

Many papers deal with the continuity (or also with stronger properties) of the map associating with any set A its center c_A, assuming it is unique. Here we are interested in estimates concerning the rate of the change: more precisely, let us substitute a bounded set A with another, say B, which has Hausdorff distance $h(A, B)$ from A; we want to give the best possible estimates for the distance between their centers c_A, c_B in terms of $h(A, B)$, and the radii of the sets A, B.

We shall try to give a complete picture concerning this problem for uniformly convex Banach spaces (or in particular, for Hilbert spaces)

The present problem has been considered in a few papers (sometimes published in journals without a large circulation: (see for example [13], but also [5] and [20])). For results in the Euclidean plane, (see for example the recent paper [7]).

We shall give a new characterization of Hilbert spaces: in fact, we will show that the best estimate we can give in Hilbert spaces cannot be expected outside this class of spaces.

Moreover, we collect, we partly sharpen and we reprove in a simple way all relevant results in this context.

We recall that X is said to be *uniformly convex*, if the following holds: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $\|x-y\| \geqslant \varepsilon$ imply $\frac{\|x+y\|}{2} \leqslant 1-\delta$. In this case, we can define the function *modulus of convexity* in this way:

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2};\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\right\} \quad (0 \leqslant \varepsilon \leqslant 2).$$

This function δ is strictly increasing in any uniformly convex space.

Let X be a Hilbert space over the real field R. Then X is uniformly convex, with modulus of convexity:

$$\delta(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \quad (0 \leqslant \varepsilon \leqslant 2).$$

Also: X is said to be *p-uniformly convex* ($p > 1$) if for some constant $k > 0$, we have:

$$\delta(\varepsilon) > k\varepsilon^p \text{ for every } 0 < \varepsilon \leqslant 2.$$

For example, Hilbert spaces are 2-uniformly convex.

Given two bounded subsets A, B of X, we indicate by $h(A, B)$ the *Hausdorff distance* between them:

$$h(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} \|x - y\|, \sup_{y\in B} \inf_{x\in A} \|x - y\|\}$$

For $x \in X$ and $r \geqslant 0$, we set

$$U(x, r) = \{y \in X;\ \|x - y\| \leqslant r\}.$$

For A bounded and $x \in X$, we shall denote by $r(A, x)$ the number

$$r(A, x) = \sup_{a\in A} \|x - a\|;$$

the *radius* of A is, by definition, the number

$$r_A = \inf_{x\in X} r(A, x).$$

A point c_A such that $r(A, c_A) = r_A$ is called a (*Chebyshev*) *center* of A. We simply speak of radius and center, instead of Chebyshev radius and center.

In uniformly convex spaces, every bounded set A has a unique center c_A; so in particular this is true in Hilbert spaces. Moreover, in the last class of spaces, $c_A \in \overline{co}(A)$, the closed convex hull of A (see for example [10], § 33).

2. Known Results

We start with a remark.

Assume that $A = \{x_1, \dots, x_n\}$ is a finite set. Let $B = \{y_1, \dots, y_n\}$ with $\|x_i - y_i\| \leqslant h$, for $i = 1, \dots, n$. Then $h(A, B) \leqslant h$.

The converse is not true. For example, let in the Euclidean plane $x_1 = (-1, 1)$; $x_2 = (-1, -1)$; $x_3 = y_3 = (0, 0)$; $y_1 = (1, 1)$; $y_2 = (1, -1)$; $A = \{x_1, x_2, x_3\}$; $B = \{y_1, y_2, y_3\}$. Then $h(A, B) = \sqrt{2}$; but $\|x_i - y_i\| \leqslant h$ for $i = 1, 2, 3$ (for any ordering of the triplets) only for $h \geqslant 2$.

But sets satisfying the "weaker" property can be slightly changed, by adding some nearby points, so that the new sets satisfy the "stronger" property. Therefore, estimates of changes related to the Hausdorff distance and proved for finite sets with the stronger property, extend to compact sets (by using an ε-*net* argument).

Some estimates concerning the "rate of change" for centers, $\|c_A - c_B\|/h(A, B)$, hold for uniformly convex spaces (or at least, for spaces with $\delta(\varepsilon)$ satisfying some particular assumptions), and spring from the consideration of the modulus of uniform convexity. Some others have been proved by using special properties of Hilbert spaces.

We recall the following result: it was proved in [4], Corollary 2.5; the "only if" part was also proved in [8], Lemma 4, then again in [11], Lemma 2.

LEMMA 0. *Let A be a bounded subset of a Hilbert space X. Then c is the center of A (if and) only if*

$$c \in \cap_{\varepsilon>0}\overline{co}\{y \in A;\ \|y-c\| \geqslant r_A - \varepsilon\}.$$

REMARK. The above result holds also in all two-dimensional, strictly convex spaces.

Already in [20], Proposition II.1.2, the following result was indicated.

PROPOSITION 1. *Let X be a Hilbert space; $A=\{x_1,\dots,x_n\}$; $B=\{y_1,\dots,y_n\}$; $\|x_i\| \leqslant R$; $\|y_i\| \leqslant R$; $\|x_i-y_i\| \leqslant h$, for $i=1,\dots,n$. Then*

$$\|c_A - c_B\| \leqslant 2\sqrt{hR+h^2}. \tag{1}$$

Still in [20], an example was given of sequences of finite sets $\{A_n\}, \{B_n\}$, in the Euclidean plane, such that $h(A_n, B_n) = n - \sqrt{n^2-m^2}$, while $\|c_{A_n} - c_{B_n}\| = m$. If we fix n and we let $m \to 0$, then this proves the following:

(NL) no general upper bound exists for $\|c_A - c_B\|/h(A,B)$.

The following estimate was given in [16], Theorem 1 and the Example at p.30.

PROPOSITION 2. *Given A, B in a Hilbert space X, we have*:

$$\|c_A - c_B\| \leqslant \sqrt{(h(A,B)+r_A+r_B)h(A,B)}. \tag{2}$$

Moreover, pairs of sets (with the same radius) exist for which (2) becomes an equality.
Clearly, (2) is an equality also when A and B are singletons.

More precisely, the above proposition was proved in [16] only for A, B compact. Giving a different proof, which makes use of Lemma 0, (2) was extended to non necessarily compact sets in [5] (the fact that compactness was not necessary was also observed in [12]). In fact, the following slightly stronger inequality can be obtained for X a Hilbert space:

$$\text{if } r_A \leqslant r_B, \text{ then } \|c_A - c_B\|^2 \leqslant (r_A + h(A,B))^2 - r_B^2. \tag{2'}$$

By refining the proof of Lemma 2.1 in [3], the following estimate for uniformly convex spaces was given in [5], Proposition 4.1.

PROPOSITION 3. *Let A, B be bounded sets and X uniformly convex; then we have*

$$\text{if } r_A \leqslant r_B, \text{ then } r_B \leqslant (r_A + h(A,B))\left(1 - \delta\left(\frac{\|c_A - c_B\|}{r_A + h(A,B)}\right)\right); \tag{3}$$

or

$$\delta\left(\frac{\|c_A - c_B\|}{h(A,B) + \min\{r_A, r_B\}}\right) \leqslant 1 - \frac{\max\{r_A, r_B\}}{\min\{r_A, r_B\} + h(A,B)} \left(\leqslant \frac{h(A,B)}{\min\{r_A, r_B\} + h(A,B)}\right). \tag{3'}$$

By using the value of $\delta(\varepsilon)$ in Hilbert spaces (recalled in Section 1), the above estimate only gives, for Hilbert spaces:

$$\|c_A - c_B\| \leqslant 2\sqrt{(h(A,B) + r_A + r_B)h(A,B)},$$

which is weaker than (2).

In case the space is p-uniformly convex, this estimate can be given in a different form: this was done in [19], Corollary 6 (see also [13]); the sharpness of the estimates was also studied.

REMARK. The following result, weaker than Proposition 3, was given in [17], Lemma 4.

Let A, B be compact and X a uniformly convex space. Then

$$\delta\left(\frac{\|c_A - c_B\|}{4[h(A,B) + \min\{r_A, r_B\}]}\right) \leqslant \frac{h(A,B)}{\min\{r_A, r_B\} + h(A,B)}. \tag{3''}$$

In fact, a careful reading of the proof shows that slight changes bring this "rough" estimate to the better estimate (3′).

A slight improvement will be given in Section 3 (Proposition 3 bis).

Still to (3′) reduces the estimate given in Theorem 3 of [19], with a proof similar to that used in [5].

If the unit ball of X lacks "good" convexity properties, then we lack in general existence and uniqueness of centers.

Some results concerning *AL-spaces* were indicated in Section 4 of [15]; for example, Theorem 4.4 there indicates that for sets with a unique center in those spaces, the change of centers when sets are perturbed can be controlled by some Lipschitz constant.

General results concerning the continuity of centers are indicated in [1], Section 6 (continuity is strongly related to the so called "quasi uniform convexity" of the space).

Also, if X is infinite dimensional and *of type* $p>1$, then the modulus of uniform continuity for $\|c_A - c_B\|/h(A, B)$ cannot be very good (see [18]); recall that being of type $p>1$ means that ℓ_1 is not finitely representable in X.

The following result was proved for example in [6] (Theorem 2.2), but rediscovered several times (see for example [9], Theorem 1). By using Lemma 0, we are giving a simple proof of it.

PROPOSITION 4. *Let X be a Hilbert space, A a bounded subset of X and c its center. Then we have*:

$$r_A^2 + \|x-c\|^2 \leqslant r^2(A, x) \quad \textit{for all } x \in X. \tag{4}$$

Proof. Let c be the center of A and $\varepsilon > 0$. Take $x \in X, x \neq c$; consider the hyperplane M through c, orthogonal to $x-c$. According to the Lemma 0, the half space determined by M and not containing x must contain some point $x_\varepsilon \in \{y \in A;\ \|y-c\| \geqslant r_A - \varepsilon\}$. This easily implies:

$$r^2(A, x) \geqslant \|x-x_\varepsilon\|^2 \geqslant \|x-c\|^2 + \|c-x_\varepsilon\|^2 \geqslant \|x-c\|^2 + (r_A-\varepsilon)^2.$$

Since $\varepsilon > 0$ is arbitrary, this proves (4). □

3. Some Refinements

Throughout this section, we shall always use the following notations. Let A, B be bounded subsets of X; without restriction, we shall assume, after a translation if necessary, that $c_A = \theta$; $c_B = c$. We set $h(A, B) = h$; $r_A = \alpha$; $r_B = \beta$; $\|c_A - c_B\| = \|c\| = \gamma$. Also, we assume that $\alpha \leqslant \beta$ (note that $\alpha + h \geqslant \beta$ always).

We slightly change (and improve) the example proving (NL) (which is contained in an unpublished thesis).

EXAMPLE 1. Let X be the Euclidean plane. Set $k = \sqrt{1+m^2}$ $(0<m<1)$. Let $A_m = \{P_{1,m}, P_{2,m}, P_{3,m}, P_{4,m}\}$, where:

$$\mathrm{P}_{1,m} = (1, 0);\ \mathrm{P}_{2,m} = (-1, 0);\ \mathrm{P}_{3,m} = \left(\frac{1}{k}, \frac{m}{k}\right);\ \mathrm{P}_{4,m} = \left(\frac{-1}{k}, \frac{m}{k}\right).$$

Now set $B_m = \{P'_{1,m}, P'_{2,m}, P'_{3,m}, P'_{4,m}\}$ where $P'_{1,m} = (1/k, (1-(1/k))m)$; $P'_{2,m} = (-1/k, (1-\frac{1}{k})m)$; $P'_{3,m} = (1, m)$; $P'_{4,m} = (-1, m)$.

We have: $h(A_m, B_m) = k-1 = \|P_{i,m} - P'_{i,m}\| (i = 1, 2, 3, 4)$; $c_{A_m} = \theta$; $c_{B_m} = (0, m)$, so $\|c_{A_m} - c_{B_m}\| = m$. Therefore: $\|c_{A_m} - c_{B_m}\|/h(A_m, B_m) = (m/k-1)$; the last quantity, for m small, can be approximated by $(2/m)$, which goes to ∞ for $m \to 0$.

We indicate a different form of Proposition 1.

PROPOSITION 1 bis. *Let X be a Hilbert space*; $A=\{x_1,\dots,x_n\}$; $B=\{y_1,\dots,y_n\}$; $\|x_i-y_i\|\leqslant h$ *for* $i=1,\dots,n$. *Let* $r_A=\alpha\leqslant r_B=\beta$; *if* c_A,c_B *are the centers of* A,B, *respectively, then we have*:

$$\|c_A-c_B\|\leqslant h+\sqrt{5h^2+2h\alpha+2h\beta}. \tag{1'}$$

Proof. We use the above notations ($c_A=\theta, c_B=c$).

Case 1. Let $\gamma\leqslant\beta-\alpha$. Then $\|c-x_i\|\leqslant\gamma+\alpha\leqslant\beta$; $\|c-y_i\|\leqslant\beta$. Therefore (1) gives:

$$\gamma\leqslant 2\sqrt{h\beta+h^2};\ \text{this implies (1}'\text{) (for example, use } 2h\beta\leqslant 2h\alpha+2h^2).$$

Case 2. Let $\gamma>\beta-\alpha$. Set $P=((\gamma-\alpha+\beta)/2\gamma)c$; then, for $i=1,\dots,n$, we have:

$$\|P-x_i\|\leqslant\|P\|+\|x_i\|\leqslant\frac{\gamma-\alpha+\beta}{2}+\alpha=\frac{\gamma+\alpha+\beta}{2};$$
$$\|P-y_i\|\leqslant\|P-c\|+\|c-y_i\|\leqslant\frac{\gamma+\alpha-\beta}{2}+\beta=\frac{\gamma+\alpha+\beta}{2}.$$

Therefore (1) gives ($\gamma=\|c\|\geqslant 0$):

$$\gamma\leqslant 2\sqrt{h(\frac{\gamma+\alpha+\beta}{2})+h^2}=\sqrt{2h\,(\gamma+\alpha+\beta)+4h^2},\ \text{or}$$

$$\gamma^2-2h\gamma-(2h\alpha+2h\beta+4h^2)\leqslant 0$$
$$\Leftrightarrow\gamma\leqslant h+\sqrt{h^2+2h\alpha+2h\beta+4h^2},\ \text{which is the thesis.}\qquad\square$$

REMARK. The proof of Proposition 1 in [20] is rather involved, but straight (no “deep” result is used); also, the estimate indicated is weak. But a careful reading of it brings again, at least for finite (then also for compact) sets, to (2).

Proposition 2 bis slightly improves Proposition 2. The method of proof is similar to that used in [17] to prove, in uniformly convex spaces, Proposition 3′.

PROPOSITION 2 bis. *Let X be a Hilbert space. Then with our notations, we have:*

$$\gamma\leqslant\sqrt{(\alpha+h)^2-\beta^2}. \tag{2'}$$

Proof. If $\gamma = 0$ then (2′) is trivial, so we assume $\gamma > 0$. We first prove a claim.

CLAIM. We always have:

$$(\alpha + h)^2 - \beta^2 + \gamma^2 \geqslant 2\gamma\sqrt{(\alpha+h)^2 - \beta^2}. \tag{*}$$

Proof of the claim. By setting $\sigma = (h+\alpha)^2 - \beta^2$ the inequality becomes $\sigma + \gamma^2 \geqslant 2\gamma\sqrt{\sigma} \Leftrightarrow (\gamma - \sqrt{\sigma})^2 \geqslant 0$, which is clearly true.

The set B is contained both in $U(c,\beta)$ and in $U(\theta, \alpha + h)$.
If $U(c,\beta) \subset U(\theta, \alpha + h)$, then:

$$\alpha + h \geqslant \beta + \gamma. \tag{5}$$

In this case
$\gamma \leqslant (\sqrt{\alpha + h - \beta})^2 \leqslant \sqrt{\alpha + h - \beta}\sqrt{\alpha + h + \beta} = \sqrt{(\alpha + h)^2 - \beta^2}$ and the result is proved.

We cannot have $U(\theta, \alpha + h) \subset U(c, \beta)$, otherwise $\beta = \alpha + h$: but then $c = \theta$, and so $\gamma = 0$.

We are in Hilbert spaces; so the intersection of the boundaries of two balls lies in a hyperplane and it is called a hypercircle (the boundary of a ball in the hyperplane).

Consider now $U(c,\beta) \cap U(\theta, \alpha + h)$, assuming that there is no inclusion: such intersection is non empty; moreover, it is a hypercircle centered at a point tc for some $t \geqslant 0$. If x belongs to the intersection, we must have:

$$(\alpha + h)^2 = \|x - tc\|^2 + \|tc\|^2$$

$$\beta^2 = \|x - tc\|^2 + \|c - tc\|^2.$$

Easy computations then give $t = ((\alpha + h)^2 - \beta^2 + \gamma^2)/(2\gamma^2)(> 0)$.
Also, the radius ρ of the hypercircle satisfies:

$$\rho^2 = \|x - tc\|^2 = (\alpha + h)^2 - \|tc\|^2 = (\alpha + h)^2 - \left(\frac{(\alpha + h)^2 - \beta^2 + \gamma^2}{2\gamma^2}\right)^2 \gamma^2.$$

If $0 < t < 1$, the ball $B(tc, \rho)$ contains $U(c,\beta) \cap U(\theta, \alpha + h)$, and then also B; thus $\beta < \rho$, and this implies

$$((\alpha + h)^2 - \beta^2 + \gamma^2)^2/4\gamma^2 = (\alpha + h)^2 - \rho^2 < (\alpha + h)^2 - \beta^2, \text{ against } (*).$$

Therefore $t \geqslant 1$: so $(\alpha + h)^2 - \beta^2 + \gamma^2 \geqslant 2\gamma^2$, which completes the proof. $\square$

REMARK 1. If equality holds in (2′) and $\gamma > 0$, we have $t = 1$: so the radius of the intersection is the same as the radius of B.

REMARK 2. It is simple to see that whenever $\alpha < \beta$, (2′) is better than (2).

REMARK 3. From (2′) (or also from (2)) we obtain:

$$\gamma \leqslant \sqrt{2h\beta + h^2}. \tag{2''}$$

REMARK 4. If A, B are balls, then (5) always holds with equality ($\alpha + h = \beta + \gamma$): (see [14]).

REMARK 5. The estimate (2′) can also be read as an estimate concerning the change of the radius, when (the amount of the perturbation, and) the distance between the centers is known. For example, $\gamma \geqslant h$ implies $\beta^2 \leqslant \alpha^2 + 2h\alpha$, or $\alpha \geqslant \sqrt{h^2 + \beta^2} - h$.

Concerning Proposition 2 bis, we note that the proof given in [12] was based on the following facts: given a bounded set A, take $\sigma > 0$; then for any $x \in X$, A contains a point a_σ such that:

$$\|a_\sigma\| > r_A - \sigma;$$
$$(c_A - x, a_\sigma - c_A) > -\sigma.$$

This is a consequence of our Proposition 4. So we can indicate another simple proof of (2′).

SECOND PROOF OF PROPOSITION 2 bis. Take $\sigma > 0$. According to Proposition 4, we have (with the usual notations), $\beta^2 + \|c\|^2 \leqslant r^2(B, \theta)$; so for some element $b \in B$, we have: $\beta^2 + \gamma^2 - \sigma < \|b\|^2$; then, by taking some $a \in A$ such that $\|a - b\| < h + \sigma$, we obtain:
$\beta^2 + \gamma^2 - \sigma < (\|a\| + \|b - a\|)^2 < (\alpha + (h + \sigma))^2$. Therefore, since $\sigma > 0$ is arbitrary, we have $\beta^2 + \gamma^2 \leqslant (\alpha + h)^2$, which concludes the proof. □

The result we are going to prove slightly improves (3) and seems to be a rather sharp estimate for uniformly convex spaces.

PROPOSITION 3 bis. *Let A, B be bounded subsets of a uniformly convex space X; $r_A \leqslant r_B$. Set $\eta = r_A + h(A, B) - r_B$; then we have:*

$$r_B \leqslant (r_A + h(A, B))\left(1 - \delta\left(\frac{\|c_A - c_B\| + \eta}{r_A + h(A, B)}\right)\right). \tag{3'''}$$

Proof. We use the usual assumptions ($c_A=\theta$; $c_B=c$; $\|c_A-c_B\|=\gamma$). Set $c'=c+\eta\frac{c}{\|c\|}$, (so $\|\theta-c'\|=\gamma+\eta$); then B is contained both in $U(\theta, r_A+h(A,B))$ and in $U(c', r_B+\eta)=U(c', r_A+h(A,B))$. Therefore, given a point $x\in B$, both θ and c' belong to $U(x, r_A+h(A,B))$: since $\|\theta-c'\|=\gamma+\eta$, the uniform convexity of X implies $[\|x-(\theta+c')/2\|\leqslant(r_A+h(A,B))(1-\delta(\gamma+\eta)/(r_A+h(A,B)))$. Therefore B is contained in $U((\theta+c')/2, (r_A+h(A,B))(1-\delta((\gamma+\eta)/r_A+h(A,B))))]$, which is the thesis. □

4. Moving Only One Point

We may wonder if moving only one point of a set can still produce, for the radius or the center of a set in case of uniqueuess, changes of the same rate as moving possibly all points of a set.

Concerning the radius, we indicate the following simple result.

PROPOSITION 5. *Let* $A=\{x_1, x_2, \dots, x_n\}$; $B=\{x_1', x_2, \dots, x_n\}$; $\|x_1-x_1'\|\leqslant h$. *Then (in any space* X*) we have* $|r_A-r_B|\leqslant(h/2)$.

Proof. Let A, B as above; also, let $A\subset U(c,r)$ for some $c\in X$ and some $r>0$. If $\|x_1'-c\|\leqslant r$, then $B\subset U(c,r)$; otherwise, if $\|x_1'-c\|=r+\eta$ for some $\eta>0(\eta\leqslant h)$, then by setting $c'=c+\frac{\eta}{2}((x_1'-c)/\|x_1-c\|)$, we clearly have: $B\subset U(c', r+(\eta/2))$; thus $r_B\leqslant r_A+\frac{h}{2}$. □

Concerning the motion of the center of a set, we consider the following example (an adaptation of Example 1).

EXAMPLE 2. Let X be the Euclidean plane. Set $k=\sqrt{1-m^2}$ $(0<m<1)$. Let $A_m=\{P_{1,m}, P_{2,m}, P_{3,m}\}$, $B_m=\{P_{1,m}', P_{2,m}, P_{3,m}\}$, where: $P_{1,m}=(k,m)$; $P_{2,m}=(-1,0)$; $P_{3,m}=(1,0)$; $P_{1,m}'=(1,m)$. We have: $\|P_{1,m}'-P_{2,m}\|$ is equal to the diameter of B_m; $\|P_{1,m}-P_{2,m}\|<\|P_{3,m}-P_{2,m}\|=2$; so: $c_{A_m}=\theta$; $c_{B_m}=((P_{1,m}'+P_{2,m})/2)=(0,\frac{m}{2})$; $r_{A_m}=1$; $r_{B_m}=\|(1,m)-(0,\frac{m}{2})\|=\sqrt{1+(m^2/4)}$. The Hausdorff distance between A_m and B_m is $1-k=(m^2)/(1+k)$. Therefore:

$$\frac{\|c_{A_m}-c_{B_m}\|}{h(A_m,B_m)}=\frac{\frac{m}{2}(1+\sqrt{1-m^2})}{m^2}=\frac{1}{2m}\left(1+\sqrt{1-m^2}\right);$$

the last quantity, for m small, can be approximated by $1/m$, which goes to ∞ for $m\to 0$.

Note that concerning the radii, we have: $|r_{A_m}-r_{B_m}|=\sqrt{1+(m^2/4)}-1=(m^2/4)\big/(\sqrt{1+(m^2/4)}+1)$; therefore, for m small, $|r_{A_m}-r_{B_m}|/h(A_m,B_m)$ is near to $(1/4)$.

5. A New Characterization of Hilbert Spaces

According to the result given in [19], Theorem 7 (see also [13]), the following is true.

For some $p>1$, let there exist $k>0$ such that, for any A and B bounded, the following estimate holds:

$$\|c_A - c_B\|^p \leqslant k((r_A + h(A,B))^p - (r_B)^p),$$

then X is p-uniformly convex.

For $p=2$ we are going to prove that in fact much more is true.

We recall the following result: (see for example [2]), Proposition (6.9‴).

PROPOSITION 6. *A Banach space has a Hilbertian norm if and only if we always have*

$$\|x-y\|^2 + \|x+y\|^2 \geqslant 2(\|x\|^2 + \|y\|^2) \text{ for all pairs } x, y \text{ such that } \|x-y\| = \|x+y\|. \tag{6}$$

As a consequence, we can give the following result.

THEOREM 7. *Let X be a Banach space. Then the norm of X is Hilbertian if and only if $(2')$ holds for every pair of sets A, B with centers.*

Proof. We assume that the norm of X is strictly convex, otherwise centers are not unique and then $(2')$ has no sense; such assumption implies that every finite set has at most one center.

If the norm of X is not Hilbertian, then there is a pair x, y, with $\|x-y\| = \|x+y\|$, violating (6): that is,

$$\|x-y\| = \|x+y\|; \quad 2(\|x\|^2 + \|y\|^2) > \|x-y\|^2 + \|x+y\|^2.$$

By changing x, y with tx, ty if necessary, it is possible to assume that $\|x\| \leqslant \|y\| = 1$; let $\|x-y\| = \|x+y\| = \sigma$ (so $2\sigma = \|x+y\| + \|x-y\| \geqslant 2\|y\| = 2$, thus $\sigma \geqslant 1$). Therefore $2\|x\|^2 > 2\sigma^2 - 2\|y\|^2 = 2(\sigma^2 - 1)$: so $\|x\| > \sqrt{\sigma^2 - 1}$ and $\sigma < \sqrt{2}$.

Now let $A = \{-y, y, x, (x+y)/2, (x-y)/2\}$; $B = \{(x+y)/2, (x-y)/2, x, x-(\sigma/2)y, x+(\sigma/2)y\}$.

Clearly, we have: $r_A = 1 (c_A = \theta)$; $r_B = (\sigma/2) (\leqslant r_A; c_B = x)$; $h(A,B) \leqslant (\sigma/2)$: therefore $\|c_A - c_B\| = \|x\| > \sqrt{\sigma^2 - 1} = \sqrt{(\frac{\sigma}{2} + \frac{\sigma}{2})^2 - 1} \geqslant \sqrt{(r_b + h(A,B))^2 - r_A^2}$, which concludes the proof. □

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